Factorization Algebras Associated to the (2, 0) Theory I

Kevin Costello
Notes by Qiaochu Yuan

December 9, 2014

The BV picture of classical field theory is that a classical field theory of dimension n sends a compact manifold M to a derived stack X with a symplectic form of degree -1. We don't need to know what a derived stack is in full detail, but in particular if $x \in X$ is a point then $T_x(X)$ is a cochain complex. The symplectic form ω then gives an isomorphism

$$T_x(X)[1] \cong T_x^*(X). \tag{1}$$

Example Let M be a 3-manifold and let $\mathrm{Loc}_G(M)$ be the moduli stack of flat G-bundles on M (the moduli of classical vacua for Chern-Simons). The tangent space at a point is

$$T_P(\mathrm{Loc}_G(M)) \cong \Omega^\bullet(M, \mathfrak{g}_P)[1] \tag{2}$$

and in particular its H^0 is $H^1(M, \mathfrak{g}_P)$, which is what one normally thinks of as the tangent space to a flat G-bundle P. Poincare duality gives an isomorphism

$$H^i(M, \mathfrak{g}_P) \cong H^{3-i}(M, \mathfrak{g}_P) \tag{3}$$

corresponding to our symplectic form.

We can do even better and think of a classical field theory as assigning to M a sheaf

$$U \mapsto \mathrm{EOM}(U) \tag{4}$$

sending an open set U to the derived stack of solutions to the equations of motion on U; these should have compatible shifted Poisson structures. If $f \in \mathrm{EOM}(U)$, then $f^*T(\mathrm{EOM}(U))$ is a sheaf of complexes on U, and we want an isomorphism

$$(f^*T(\mathrm{EOM}(U)))^\vee \cong f^*T(\mathrm{EOM}(U))[1]. \tag{5}$$

Here $^\vee$ denotes the Verdier dual.

Example In Chern-Simons, $\mathrm{EOM}(U) = \mathrm{Loc}_G(U)$. Poincaré duality now says

$$H^\bullet(U, \mathfrak{g}_P)[1] \cong (H_c^\bullet(U, \mathfrak{g}_P))^* \tag{6}$$

where H_c denotes compactly supported cohomology.

Example In holomorphic Chern-Simons, let M be a Calabi-Yau 3-fold. $\mathrm{Bun}_G(M)$ (where this now denotes holomorphic rather than flat bundles) has a shifted symplectic pairing coming from Serre duality

$$T_p(\mathrm{Bun}_G(M)) \cong \Omega^{0,\bullet}(M, \mathfrak{g}_P)[1] \cong (\Omega_c^{0,\bullet}(M))^*. \tag{7}$$

This is a weak equivalence of cochain complexes of topological vector spaces.

Example In a mixed example, we can look at bundles on $\mathbb{C}^2 \times \mathbb{R}$ which are holomorphic on $\mathbb{C}^2$ and flat on $\mathbb{R}$.

Q: can you elaborate on what a shifted Poisson structure is?

A: let X be a derived thing. Locally, its functions $O(X)$ look at least sort of like a dg algebra. A shifted Poisson structure on $O(X)$ is a Poisson bracket that sends two elements $f, g \in O(X)$ of degrees $\deg f$ and $\deg g$ to an element of degree $\deg f + \deg g + 1$ satisfying appropriate versions of the usual axioms, with various signs inserted.

Example Consider Rozansky-Witten theory. Let X be a holomorphic symplectic manifold (although we won't use the holomorphic structure here) and let M be a 3-manifold. We'd like to send M to the derived stack of locally constant maps from M to X, which we'll write as

$$\mathrm{Maps}(M_{dR}, X). \tag{8}$$

The tangent space at a map $f : M \to X$ is

$$T_f(\mathrm{Maps}(M_{dR}, X)) = \Omega^\bullet(M, f^*(TX)). \tag{9}$$

Given $\alpha, \beta \in \Omega^\bullet(M, f^*(TX))$ we can take

$$\omega(\alpha, \beta) = \int_M \omega_x(\alpha, \beta) \tag{10}$$

where ω_x is the symplectic form on X, and this is a symplectic form of degree -3. We wanted a symplectic form of degree -1, but Rozansky-Witten theory is only $\mathbb{Z}_2$-graded anyway, so that's okay. One way to say this is to introduce a parameter $\hbar$ of degree -2 and to look at $\frac{\omega}{\hbar}$.

Q: can we do this for symplectic forms of any degree?

A: it should have odd degree.

Q: why is it important that $\hbar$ is even?

A: it should be central.

Q: what happens if M is a 1-manifold instead of a 3-manifold?

A: we get topological quantum mechanics, with a Hamiltonian of zero.

Now suppose V is a complex symplectic vector space and G is a complex algebraic group acting on V by symplectomorphisms. Then we can construct the symplectic reduction $V//G$, but we'd like to do take this reduction as a derived stack. For example, when $V = 0$ we will get the cotangent bundle $T^*(BG)$, and in general the tangent space at the identity is

$$T_0(V//G) = \mathfrak{g}[-1] \oplus V[0] \oplus \mathfrak{g}^*[1]. \tag{11}$$

One of these interesting extra components comes from the moment map equation and the other comes from remembering stabilizers.

We can define Rozansky-Witten theory with target $V//G$, and this is one of two topological twists of 3d $N = 4$ gauge theory. All of the field theories I know how to write down in dimensions less than 6 I know how to write down as variations of this.

What is twisting? In 3d $N = 4$ SUSY, we extend the Poincaré Lie algebra by an 8-dimensional odd vector space

$$S \otimes \mathbb{C}^4 \tag{12}$$

where S is the fundamental representation of $\mathrm{SL}_2(\mathbb{C}) \cong \mathrm{Spin}_3(\mathbb{C})$. Choose $Q \in S \otimes \mathbb{C}^4$ such that

$$[Q, Q] = 0. \tag{13}$$

Then twisting of a field theory means taking Q-cohomology of everything in the field theory. Or, better: if the theory has some BRST differential Q_{BRST} then we replace it with $Q_{BRST} + Q$.

This makes everything much simpler. Let's choose $Q = (Q_1, iQ_1, 0, 0)$. Then $Q^2 = 0$. Operators in the image of $[Q, -]$ are trivial in the twisted theory. The image of $[Q, -]$ in complexified translations $\mathbb{C}^3$ turn sout to be 2-dimensional, and the result is that the twisted theory is topological in one direction x_1 and holomorphic in the remaining two directions x_2, x_3.

What do we get? The classical solutions on $\Sigma \times S^1$ are maps

$$T[1]\Sigma_{\bar{\partial}} \times S^1_{dR} \to V//G \tag{14}$$

and the point is that functions on $T[1]\Sigma \times S^1_{dR}$ is

$$\Omega^{\bullet,\bullet}(\Sigma) \otimes \Omega^{\bullet}(S^1) \tag{15}$$

with differential $\bar{\partial}_\Sigma + d_{dR}$. Serre duality implies that this is a field theory, and the claim is that this is the twist as above.

For example, on $\mathbb{C} \times \mathbb{R}$ and with trivial gauge group, the theory is free and the tangent complex (fields) is

$$\Omega^{0,\bullet}(\mathbb{C}) \otimes \Omega^{\bullet}(\mathbb{R})[\varepsilon] \otimes V \tag{16}$$

where $|\varepsilon| = 1, \varepsilon^2 = 0$. This is a twist of the free hypermultiplet.

There are two further topological twists. They come from extra odd symmetries $\frac{\partial}{\partial \varepsilon}$ and $\varepsilon \frac{\partial}{\partial z}$. The second one turns on the de Rham differential, gets us Rozansky-Witten theory, and is called the B-twist. The first one is called the A-twist, and in it we consider maps $\mathbb{C} \times \mathbb{R} \to V//G$ which are holomorphic in $\mathbb{C}$ and constant in $\mathbb{R}$.

3d mirror symmetry exchanges A-twists and B-twists. The A-twist is going to be related to the Coulomb branch and the B-twist is going to be related to the Higgs branch.